Digital geometry and its applications to computer vision


Digital geometry is the study of geometric properties of discrete point sets that are considered to be digitized objects in digital images. Since any images treated by computers are digitized, we cannot avoid influences of such digitization upon the geometric properties. For digitized objects, there exist particular properties caused by the discreteness and the finiteness of digital images, and they often provide efficient algorithms for computing geometric operations on digitized objects. Another positive aspect of digital geometry is that it implies the exact computation in the integer domain, so that the obtained results are guaranteed against computation errors; no digitization error exist neither since the geometric models are directly designed for discrete point sets whose coordinates are only integers.

On the other hand, computer vision, which is a conceivable application of digital geometry, is the study of extracting scene information from digital images. Conventionally its techniques use geometric models defined in the continuous framework while computations are digitally performed, so that the obtained results always involve digitization errors and computation errors. Accordingly these errors violate geometric properties that the object originally has and the obtained results are not reliable. In order to overcome such a serious problem, we study digital-geometrical models that allow us reliable geometric computing for discrete point sets.

The main results of our work on digital geometry for computer vision applications between 2008-2009 are as follows:

**Discrete epipolar geometry:** In order to directly deal with pixels considered as square units, instead of points, in images, we established, with A. Sugimoto, the discrete epipolar geometry, a discrete version of the epipolar geometry, which gives geometric relationships between pixels in two different views. In this framework, we proposed a method for determining the discrete epipolar line, which is defined by a pair of inequalities, together with a method for identifying the corresponding region in a third view. Differing from the conventional approach, our proposed methods exactly identify how digitization errors propagate into the epipolar line and into the corresponding region in a third image.

**Discrete rotations:** Rotations of digital images are required in many applications. Given a 2D digital image and its rotated image, there exist many angles giving the rotation between them. With Y. Thibault and A. Sugimoto, we proposed a method for obtaining the set of such admissible rotation angles by using hinge angles, introduced by B. Nouvel and E. Rémila, whose principal operation, comparison, is made only with integer computations. We also succeeded in extending their notion for 3D digital images so that all rotations of 3D digital images can be realized by using only integers.

**Digital planar surface segmentation:** For segmenting a digitized surface into digital planar surfaces, geometric features such as normal vectors, which are generally estimated from the neighboring points, are used. The obtained results are thus sensitive to digitization errors. With L. Buzer, I. Shimizu and A. Sugimoto, we proposed a method for the segmentation by using the finite number of local geometric patterns that appear on digital planes, without estimating normal vectors.

(a) (b)

(a) Discrete rotations: upper and lower bounds of rotation angles between two sets of corresponding pixels. (b) Digital planar surface segmentation: each color corresponds to a planar part except for green whose points have non-linear local geometric patterns.

**Publications:** journal article [21], conference articles [103, 100, 102, 186].
Discrete topology


This theme is focused on the study of discrete/combinatorial topology. Our theoretical framework is mainly based on complexes. In this context, an object is a set of faces of different dimensions.

Our main results are the following:

- We propose new characterizations of simple points in 2D, 3D, and 4D discrete spaces, a simple point of an object is a point which can be removed from the object without altering its topology. Thus, these characterizations may be used for improving the efficiency of all algorithms which preserves the topology of objects (homotopic transforms). We also characterize 3D minimal simple pairs.

- We investigate the notion of critical kernels, critical kernels provide a sound basis for the study of operators which remove a whole subset of an object (and not only a simple point) while preserving its topology. From this notion, we proposed a wide variety of 2D and 3D parallel thinning algorithms which allow to extract different kinds of skeletons of an object. We also show that critical kernels are a non-trivial generalization of all previous related notions, in particular the notions of minimal non-simple sets and P-simple points.

- We propose a characterization of loops in n-dimensional discrete toric spaces. This characterization leads to a linear time algorithm which detects the very loops in a given object (inside a torus) which cannot be continuously deformed to a point. Some applications, such as the simulation of fluid flows in porous materials, rely on 3D toric spaces and require the detection of such loops.

- We establish a deep link between the notion of a watershed and discrete homotopic transforms. For that purpose we introduced a definition of a watershed of a map defined on a pseudo-manifold. We showed that any watershed of such a map may be extracted from a homotopic thinning of this map.

Geometric pattern matching


This work was done while D. Aiger was with Ben-Gurion University, Israel.

The problem of Geometric Pattern Matching (GPM) is that of searching for a given geometric pattern in a scene composed of geometric objects. There are many variants of this general problem depending on the nature of the geometric objects in the pattern and the scene, the nature of the resemblance (similarity) measure, and the rules of the search. Geometric objects may be points, lines, or polygons, in two-, three-, or d-dimensional space. Resemblance between patterns is a qualitative measure determined by the nature of the problem. Once the nature of resemblance is decided, an appropriate resemblance measure has to be proposed. Rules of the search determine what class of transformations the pattern is allowed to undergo during the search for a match.

I present efficient algorithms to match sets of points in the plane and in high dimensions. The problem appears in a large number of applications in many fields such as computer vision, computational biology, computer graphics and more. I investigate the problem from the theoretical point of view by providing upper bounds on the runtime of the proposed algorithms in the worst case, and also give practical solutions that can be effectively applied in real problems. The work is a collection of five articles which have appeared in journals during the study.

In the theoretical part, I present new efficient algorithms to match sets of points in the plain under similarity transformations (rotation, translation and scale), a problem that was left almost untreated previously. Previous solutions usually require long runtimes and are not practical. My algorithms make extensive use of powerful tools, randomization and approximation, to provide runtime which is nearly quadratic in the size of the input. In another work I present new algorithms where the runtime depends on the configuration of the input and not only on its size (input/output sensitive algorithms) and which achieve excellent performance in most cases. For a problem in Computer Graphics I illustrate the idea of using geometric invariants on registration of two pointsets in 3D (Figure 1). This allows improving existing algorithms by an order of magnitude. I expand the idea further and present new algorithms for points in high dimensions that work under every transformation.

The ideas presented in the thesis are realized for a number of applications. I show effective implementations of these ideas using parallelization which is provided by standard graphics processors (GPU). These implementations are very fast for practical data (Figure 2).

Figure 1. Reconstruction from raw scans using 4-points congruent sets. Reconstruction results from nine input scans of a shiny water jug. Neighboring scans have 40% overlap or less, and required an average of 16 seconds for fully automatic alignment starting from arbitrary initial poses. Pairwise alignment results are robust even with low overlap. Typical results are shown in (a) and (b), where for visualization we roughly mark the overlap regions in blue. The final alignment result, (c) and (d), is obtained without any data smoothing, outliers removal, local ICP refinement, global error distribution, or any assumption about starting alignment.

Figure 2. Model based object recognition problem for rigid transformations: (a) input image (499 oriented points), (b) model (30 oriented points, magnified in the drawing), (c) edge map, (d) detection results where the polygonal model is marked. The runtime of the whole recognition, edge detection + matching, was done in 25 milliseconds while the matching alone took 10 milliseconds.

Publications: journal articles [25, 26, 58, 59].
Geometric simplification


We study the conversion between Euclidian curves and discrete geometric objects. Our approaches link problems from different fields like computational geometry, discrete geometry, pattern recognition and digital geometry. Our aim is to propose new algorithms designed to reach a linear or quasi linear time complexity. Fundamental algorithmic notions from optimization, data structure, graph theory are used to achieve such efficiency.

Our main results in 2008-2009 are related to the simplification of simple polygonal chain (see Figure). The problem of approximating such an object arises in many applications including geographic information systems (GIS), cartography, computer graphics, data compression or medicine. Enhancing the quality of conversion has been recognized as an important factor in advancing the research in this field.

As an application, we propose the first approximation criterion which entails to control the quality of the rendering of the simplified chains. Our main contribution consists of setting up the first near-linear time algorithm that ensures optimality in the number of segments and that retains the shape of the input polygonal chain at the same time. Thus, our approximation criterion guarantees that the original chain lies at most half a pixel away from the rendering of the simplified chain.

With Emilie Charrier, we consider the problem of finding the integer points on a regular grid whose associated polygonal chain best approximates a given Euclidian straight line. We combine existing results from number theory based on the Klein’s sails with tools from computational geometry based on the computation of convex hulls to obtain a new generation of algorithm. In fact, the overall worst case time complexity is equivalent to previous algorithms but, our approach is the first being output sensitive. Moreover, the resulting algorithm is very simple and so is suitable for implementation.

With F. Feschet and A. Faure, we study the simplification of closed polygonal chains. Previous problems were linked to an open chain connecting \( n \) points \((p_1, \ldots, p_n)\). By applying previous methods, we would solve the \( n \) subproblems obtained by considering the optimal simplification of an open chain starting at \( p_i \) where \( 1 \leq i \leq n \). We would obtain the global solution by selecting among the \( n \) simplifications the one with a minimum number of points. Such an approach would increase the time complexity by a factor \( n \). Thus, by carefully analyzing the results built during the subproblems, we succeed in proving that we can efficiently use the calculations operated from the \( k^{th} \) point to build the simplification of the chain starting at \( p_{k+1} \). Such a way, we can achieve a quasi linear time complexity.

Illustration of the simplification of simple polygonal chain.

Publications: journal articles [41, 19, 40].
Topological operators for image processing


We follow an approach based on discrete structures (graphs, finite topologies and orders . . .) to define new operators for image analysis and image processing. More precisely, we study topological and morphological notions in order to design new algorithms that prove useful in image processing. Fundamental topological notions such as adjacency, connectivity, frontier, which are central to this approach, constitute indeed a basis for the processing of spatial data.

A major part of our work is dedicated to the study of binary image transformations that preserve the topological characteristics of an image. These transformations are based either on the notion of simple point, well-known in digital topology, or on the collapse operation, from the field of algebraic topology.

Our main results in 2008-2009 are related to parallel homotopic thinning procedures, and to skeletons based on the Euclidean distance.

In the framework of critical kernels introduced by G. Bertrand, we have proposed new parallel homotopic skeletonization algorithms in 2D and 3D spaces, which are significant advances with respect to the state of the art. Thanks to the main property of critical kernels, and to the huge flexibility induced by this property, we were able to design many different algorithms that meet particular geometric constraints, both in 2D and 3D: curvilinear or surface skeletons, preservation of the medial axis, invariance w.r.t. some rotations, etc.

With Nicolas Passat, we have studied the notion of minimal simple pair and proposed a thinning algorithm that detects and removes such pairs. This algorithm is less likely than classical ones to get blocked in pathological configurations.

With A. V. Saúde, J. Chaussard and H. Talbot, we proposed some methods to produce different kinds of skeletons which are centered in the original object with respect to the Euclidean distance. In particular, we introduced a digital version of the $\lambda$-medial axis, a filtered “continuous” skeleton which exhibits good stability w.r.t. noise. We also proposed new thinning procedures in cubical complexes, which allow us to produce skeletons having desired dimensionnality (e.g., purely 2D or 1D skeletons of 3D objects).

We participated to several projects in different application domains (medical image analysis, material image analysis, virtual reality), which allowed us to apply and validate these new operators.

Publications: journal articles [18, 22, 43], conference articles [101, 173, 180, 181, 182, 169].

(a): A 3D object $X$. (b): Values of $\lambda$ (the filtering criterion) at all points in $X$, displayed using a “heat” color code. (c): A $\lambda$-medial axis of $X$ (threshold of b for a given $\lambda$).